Chapter 3

The New Math and Some Physics

Bruno considered geometry fine for children but a waste of time for adults. He had a vague sense that its self-imposed limitations hindered science. He was far more progressive in his science. He advocated the same physical laws for the Earth and the heavens. He considered motions to be relative, emphasizing that the Earth moves. Yet he did not envision that increasing the scope of mathematics would unleash the power of relativity and unified physics.

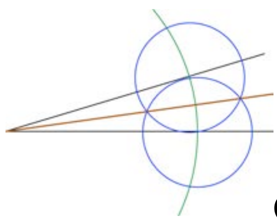
Relativity of motion

We are used to thinking as the Earth as fixed, terra firma. Only a tort lawyer might say that the defendant's barn struck his drunken plaintiff's car. This natural view of absolute motions retarded the progress of medieval science. A simple experiment is to play catch in a moving car or plane. If you do not look outside, you cannot tell how fast the car is moving from the game of catch. You can only tell when the velocity of the car changes. The ball flies forward when the driver brakes.

Both Bruno and Galileo used a moving ship as their example of the relativity of motion. They noted that a weight dropped from the mast falls in the same place whether the ship is moving or not. Their point is that, like the passengers of a ship, the inhabitants of the Earth experience no sensation of the movement of the Earth. Galileo took it one step further; one can simply add up velocities. The velocity of the weight observed on the shore is the velocity of the ship plus the velocity of the weight as observed from the ship.

Galileo was correct to the level that we can observe in our daily lives. We have already seen that the motion of the Earth in its orbit causes the aberration of starlight. Other subtle effects became evident with developments in mathematics and physics.

Flexible mathematics



(Bisection: D3.1) In classical geometry, one is allowed to have a compass and a straight edge. These are assumed to work perfectly so that lines have no width. Proofs are exact, but they are limiting. One can easily divide an angle in two, but it is impossible to divide an angle into three equal parts. One can get very close with repeated bisections, but never exact.

One is limited to straight lines and circles. Being limited to circles straight-jacketed Copernicus. The orbits of the planets are in fact not circular and he still needed a morass of epicycles to describe them. Galileo used circular orbits but made no detailed calculations of planetary position.

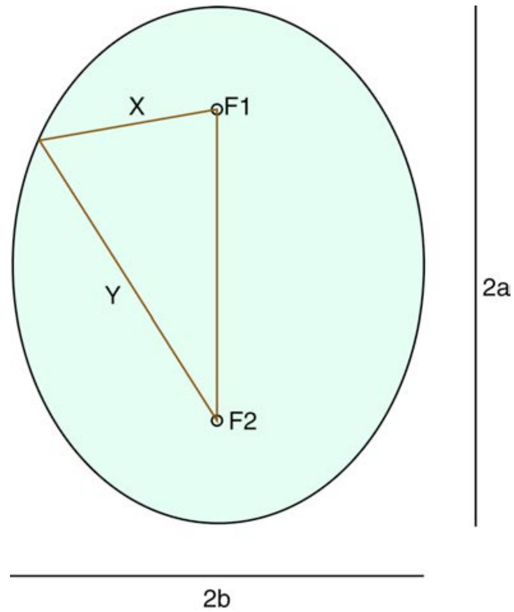


Figure 1: An ellipse has major axis *a* and minor axis *b*. The sum of the distances *X* and *Y* from the foci *F1* and *F2* remains constant. This lets one draw an ellipse with a loop of string around two tacks. This method was quite messy with ink in the days before computer plotting. See Primer at the end of the chapter for more on ellipses.

Kepler and ellipses. Kepler abandoned the assumption of circular orbits by using ellipses to describe oval orbits. One may draw an ellipse with a loop of string and two pins (Figure 1). The length of the loop stays constant as one goes around. Mathematically, the two pinpoints are the foci (plural of focus) of the ellipse. The sum of the distances from each focus to a point on the ellipse stays constant. A circle is an ellipse with both foci in the same place.

Kelper found out from analysis of reams of data that:

(1) The orbits of the planets are ellipses with the Sun at one focus.

(2) The planet sweeps out an equal area with lines going to the Sun over equal times. (See Figure 2)

(3) The square of the period (year) that a planet orbits the Sun is proportional to the cube of the long axis of the ellipse.

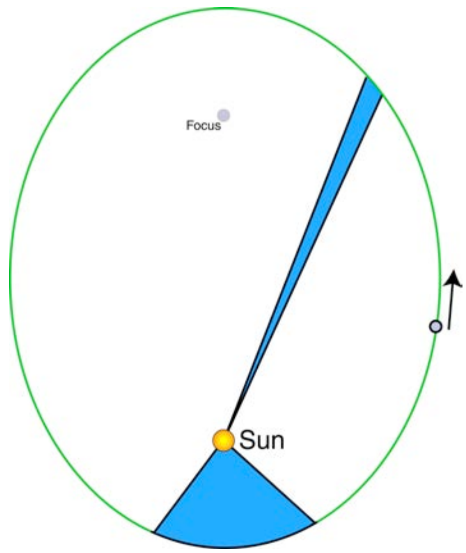
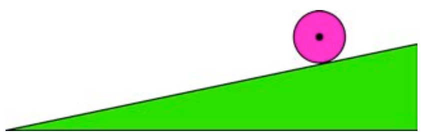


Figure 2: Kepler's second law is easily visualized geometrically. A planet orbits in an ellipse with the Sun at one focus. The other focus has only mathematical significance; a spacecraft visiting it would find nothing special. The planet sweeps out equal shaped areas (shaped) in equal intervals of time. It thus moves fast when it is near the Sun.

Unlike epicycles, these laws are predictive. One can see if a newly discovered comet, for example, follows them. The subtle deviations of the actual planetary orbits from these laws provided the keys to the development of physics. A more general way of representing curves was needed to do this.

Graphs and 3-dimensional curves. René Descartes (His name translates "Of the maps.") provided the tool, which may seem mundane to us. In two-dimensions, one represents positions, like a map. The coordinates might be north and east. In three dimensions, one needs three perpendicular coordinates, north, east, and up would do locally. One can pick the origin wherever one chooses, usually for convenience. One can move back and forth between coordinate systems with different origins and different coordinate directions. The full power of relativity comes from requiring that physical laws not change when one changes coordinate systems.

With regard to astronomy, one picks 3 perpendicular coordinate directions on the celestial sphere of fixed stars. Putting the Sun at the center, one can represent the position of each planet with these three coordinates. One needs only 5 parameters to describe the orbit of each planet with Kepler's laws, three to describe the long axis of the ellipse and two to describe the short axis (which is perpendicular to the long axis). One more is needed to give the planets position on its orbit. This is a lot more elegant than a myriad of epicycles.



(D3.2) Falling bodies and calculus. In the meantime, Galileo had been studying the dynamics of falling bodies. To slow them down, he cleverly let them roll down a ramp. He found that a roller that went (to make it easy in modern units) 1 meter by the end of the first second from rest, went 4 meters by the end of the next second, 9 meters by the end of the third, and so on. The distance traveled was proportional to the time squared (Figure 3). The average velocity over each second increased linearly as 1, 3, 5 meters per second. In three dimensions, he found that the vertical velocity of a dense object in flight decreased (with upward velocity positive) linearly with time. The horizontal velocity stayed more or less constant. For a real cannon ball, one had to correct some for air resistance.

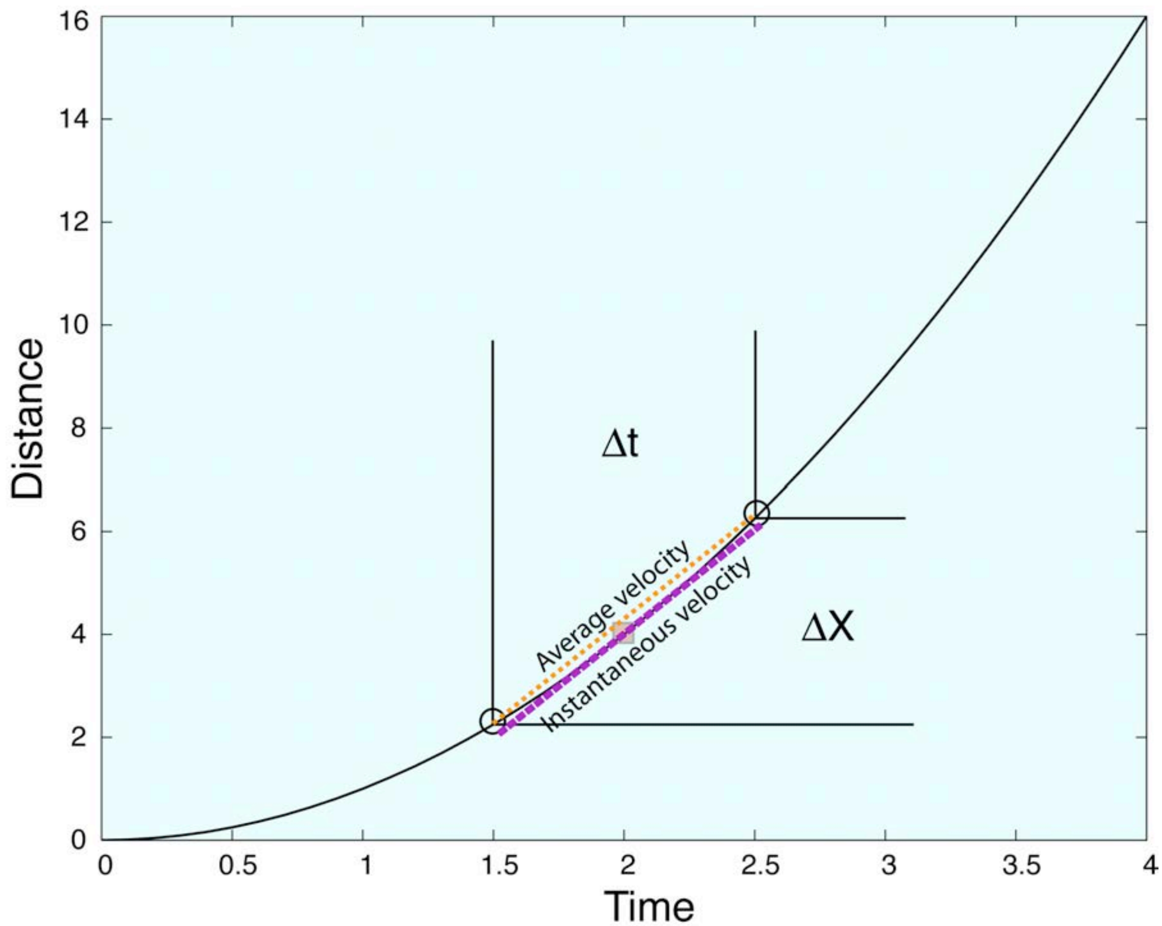


Figure 3: Graphs let one see the results of physical experiments and here simple calculus. The distance traveled by a roller on a ramp is a function of the time since it started from rest. The object moves a distance ΔX of 4 meters between 2.25 and 6.25 m in time Δt of 1 second between 1.5 and 2.5 seconds. The average velocity is the slope of the line between the two points on the curve. We obtain the derivative of distance with respect to time (that is the instantaneous velocity at 2 seconds) by making the intervals very small. In this special case, the instantaneous velocity is the same as the average velocity.

Galileo and Kepler were not particularly receptive to each other's innovations. (It would have been dangerous for Galileo to openly support Kepler, a Protestant, by name.) Galileo retained the medieval notion of circular motions being natural in the heavens. He did not apply his laws of falling bodies to the planets. As legend has it, a falling apple jolted Isaac Newton into doing just that.

Newton's contribution involved the instantaneous change in quantities. I illustrate the method by graphing the results of the ramp experiment (Figure 3). My intent is to illustrate what calculus does without having to teach it. You can see the objectives of higher mathematics without being facile with them. This is much like art and music appreciation.

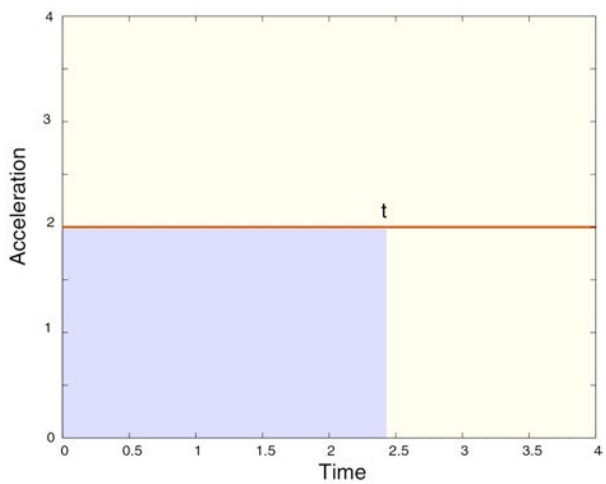


Figure 4: This graph illustrates simple calculus. The acceleration of the object on the ramp is constant, 2 meter per second per second. The velocity at time *t* is proportional to the blue shaded area of the rectangle.

The distance that the roller traveled on Galileo's ramp increased with the square of time. Neither Galileo nor Newton had a good way to measure instantaneous velocity. Rather they had to measure the position of the roller at 2 times. This lack of equipment led to the key concepts of calculus.

For example, say that we want the instantaneous velocity at 2 seconds. We could start with measuring the position of 1 meter at 1 second and 9 meters at three seconds. In the notation of calculus, we let the variable t represent time and the variable X be the distance traveled along the ramp. Mathematicians use the Greek letter Δ to indicate the change over our interval. That is the time interval (here 2 seconds) is called Δt and the distance interval is ΔX. The average velocity is

(9 meters – 1 meter) $\div$ (3 seconds – 1 seconds) = $\Delta X \div \Delta t$

which is 2 meters per second. We can make the time interval Δt smaller say from 1.5 to 2.5 seconds. The average velocity is

(6.25 meters – 2.25 meters) ÷ (2.5 seconds – 1 seconds)

which is again 2 meters per second. If we made the time interval Δt smaller (or larger), the average velocity centered on the time of 2 seconds would still be 2 meters per second. The reader who is familiar with algebra can quickly prove this fact. For more complicated motions, the average velocity over an interval does depend on the length of the interval.

Continuing with calculus, we define the instantaneous velocity to be the average velocity when the time interval Δt becomes extremely small. Graphically, the quantity is the slope of the curve of position versus time curve in Figure 3.

We have already seen that the average velocity (over a second) increases linearly with time. The instantaneous velocity also increases linearly with time. The area under the graph of the velocity versus time gives the distance traveled. Remembering that this area is one-half of the base times the height, we find that the distance traveled is proportional to the square of the time. We can also plot the slope of the velocity curve, that is, the change of velocity per time, called the *acceleration*. In this experiment, it is constant. Galileo got to this point.

In modern terminology, the velocity is the derivative dX/dt where the d's replace the Δ's to indicate that the intervals are arbitrarily small. Similarly, the acceleration is the derivative of the velocity with respect to time, dV/dt, which is here 2 meters per second each second, or more compactly, 2 meters per second squared (Figure 4).

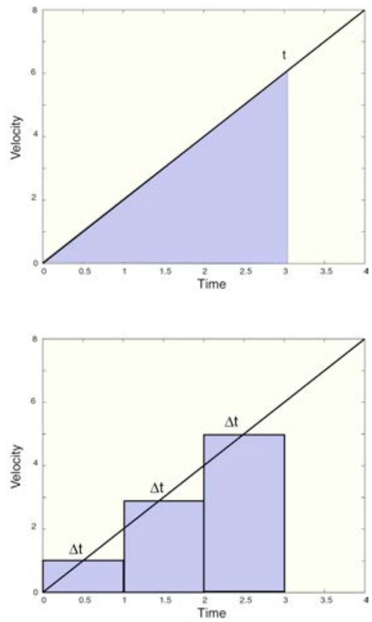


Figure 5: More simple calculus. The velocity of the object increases linearly with time (top). The distance traveled at time *t* is proportional to the blue shaded area. We can compute the blue shaded area by dividing up into interval rectangles of length Δt. In calculus, one assumes that the intervals are very small.

We can get the position if we know the history of the velocity. Distance equals rate times time. To do this we add up the change of position for each time interval (Figure 5). That is,

$$\text{Final position } = \text{ starting position} + \sum V_i \times \Delta t_i$$

where the sum sign capital sigma Σ tells us that we need to add up the product of the velocity and length of each time interval, which I have denoted by a subscript i. In calculus, the intervals are assumed to be very small and the sum is replaced by an integral. The position at some time f after the starting time defined as zero 0 is:

$$X(f) = X(0) + \int_0^f V \; dt$$

The differential d*t* reminds us that we are taking time steps. The integral sign $\int$ (the funny s in books printed before about 1800) tells us that we need to add up velocity steps between the starting time 0 and the final time *f*.

Calculus is extremely useful in all quantitative science. Scientists represent physical laws in terms of instantaneous change in time and space, called a differential equation. They can predict the subsequent behavior of a system from its initial conditions with their differential equation. They also have rules, like the conservation of energy, that involve adding up everything in their system. They use integrals to do this.

In Newton's case, all he needed to get the positions of the roller in one dimension or of a planet in three dimensions was a rule to determine the velocity at each time. Using the calculus that he just invented and simple special cases like the roller, he proposed:

(1) Every body at rest stays at rest and every body in motion stays in straight motion unless it is acted on by a force. (This rid the world of the medieval concept that circular motion was natural in the heavens and the impetus concept that force was needed to keep a body moving. It also included the concept that velocity and rest are relative. See note on impetus at the end of this chapter.)

(2) The acceleration of a body is inversely proportional to the body's mass and directly proportional to the forces acting on it. (The first law is the special case of the second law where there are no forces. The weight that we measure with a scale with springs is a force. The balance in a doctor's office measures mass by comparing your

mass to a known mass.)

(3) Every action is opposed by an equal and opposite reaction, or there are no unbalanced forces. (For example, the force of water on oars balances the force the oars exert on the water. The forces acting between two bodies in isolation are equal and opposite so no net force acts on the two bodies. These forces do not accelerate the center of mass of the two bodies.)

Newton proposed the law of universal gravitation to complete his physical description of planetary orbits. The gravitational force between two bodies acts along the line between the two bodies and is proportional to the product of their masses and inversely proportional to the square of their distance of separation.

The forces are equal and opposite as in the third law. Equivalently the force per mass on one object is proportional to the mass of the other and the inverse square of the distance. The acceleration of the first body is proportional to this quantity because of the second law. The acceleration from gravity decreases with the square of distance just like the brightness of a light from a star in Chapter 2.

Using calculus, Newton derived Kepler's laws with the correct assumption that the Sun has almost all the mass of the solar system. Derivation of the elliptical orbit is complicated, but the other two laws have simple illustrations that can be derived with brief mathematics that is not shown here. Kepler's second law of equal areas is equivalent to saying that gravitational force acts on a line between the Sun and a planet. It also implies the conservation of angular momentum, defined mathematically from the product of the instantaneous velocity and the distance from the Sun. You have probably seen the effect of conservation of angular momentum when ice skater spins more rapidly after she pulls in her arms. The reader who is familiar with geometry can quickly show that the graph in Figure 2 implies conservation of angular momentum.

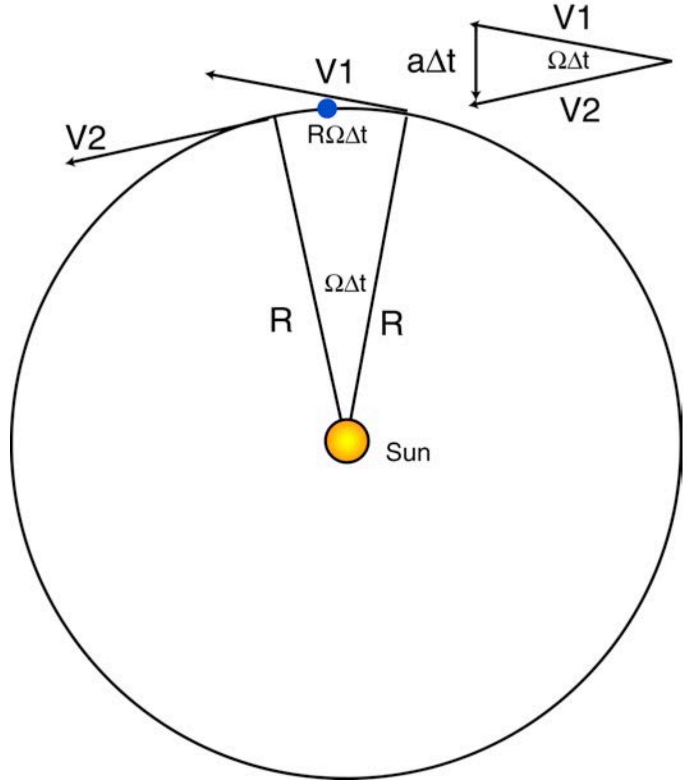


Figure 6: Graphical demonstration of Kepler's third law. The planet orbits in a circle of radius R at an angular rate Ω. Its speed remains constant, but the velocity changes over the interval t. Note the change in velocity, the acceleration a is in the direction between the planet and the star.

Kepler's third law also has a simple graphic derivation for circular orbits (Figure 6). This requires some algebra. To allow myself to use the small angle formula, I express angles in radians (see small angle formula in last chapter) and the rate of rotation in radians per second, which I denote by Ω. The planet covers an angle Ω times Δt over the small time interval Δt. The speed (magnitude of the velocity) S does not change as the planet orbits, but the direction of the velocity direction does. Using the small angle formula again, the change in velocity is the speed *S* times the angle $\Omega \times \Delta t$. Note that the change is radially inward toward the Sun. The gravitational force is proportional to the product of the mass of the Sun *M* times the gravitational constant *G* and inversely proportional to the square of the planet-Sun distance *R*. Once again using the small angle formula, the speed is $\Omega \times R$. Equating the force per mass to the acceleration gives the equation

$$\frac{\Delta V}{\Delta t} = \frac{V\Omega\Delta t}{\Delta t} = \Omega^2 R = \frac{GM}{R^2}$$

The planet revolves 2 times π radians revolution so its year *Y* is 2 times π divided by the rotation rate Ω. We get Kepler's third law in terms of the radius making this substitution in the acceleration equation

$$R^3 = \frac{GMY^2}{(2\pi)^2}$$

One does not need to know the mass of the Sun to apply this equation, only the product *GM*. In fact, the gravitational constant is difficult to measure because it is hard to get a big enough mass in the laboratory to produce a precisely measurable gravitational force. The constant was not measured reliably until the early 1800s. It is still the least accurately measured physical constant.

The Moon provided the start to Newton's thinking because its orbit is mainly determined by the mass of the Earth. Newton got the Moon's orbital acceleration from the length of the month and he knew the acceleration of gravity at the Earth's surface. He knew the radius of the Earth's surface and the distance to the Moon from the Earth's center. With a quick calculation (left here for the reader familiar with geometry), he confirmed that the inverse square law for gravity is basically correct. This simple calculation conclusively unified astronomy with physics.

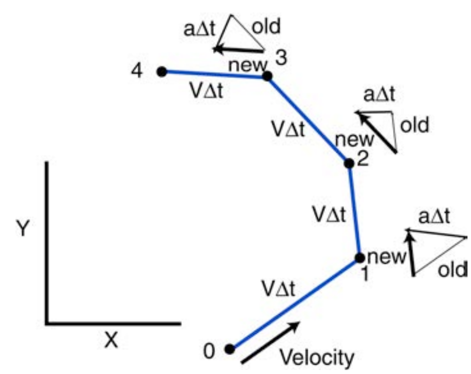


Figure 7: Graphical illustration of numerical calculation of planetary orbit. At time 0, the velocity is known. One obtains the position at time 1 by moving $V\Delta t$. One obtains the forces at the new position and changes the velocity by the acceleration times the time interval, mathematically $a\Delta t$. One gets the position at point 2 using the new velocity. One continues alternately updating velocity and acceleration. Small time and space steps are needed for the procedure to work accurately. This is a limitation even with modern computers. With calculus, it is sometimes possible to do the calculation with simple functions, like for a planet in circular orbit.

Newton had a general method for describing the motions of the planets without using his new calculus (Figure 7). One needs only to add up the forces per mass on one planet from the Sun and the other planets and use this acceleration. The acceleration acts over the time interval Δt changing the velocity. One then does this for the Sun and all the other planets to assign new velocities. One then uses the computed velocities to get new positions after time Δt and then repeats getting new velocities. The procedure works if one takes small enough time steps. This was a major problem with hand calculations in

Newton's time and still a problem if one wishes to model orbits over millions of years on a computer. Mathematical astronomers have put considerable effort into using the fact that Kepler's laws are approximately correct to make calculations more efficient.

Unanticipated results from mathematical physics

If we proceeded with the planets as point masses as discussed above, we would find out that our predictions are not perfect. Some of the imperfection occurs because we do not have exact estimates of the masses and the starting conditions. If we have been careful in estimating errors in our data, it is a simple though cumbersome matter to predict how accurate our predictions should be. Typically the actual inaccuracy of predictions using point masses exceeds the expected errors. Because we have a physical model, we can learn a lot from these discrepancies. With epicycles, we would be able to do nothing but add still more epicycles and tweak the ones we already have.

Precession and related effects. We would find that the orbital distance of the Moon increases with time and on average the rotation of the Earth slows down. This is due to tides that dissipate energy. The net effect is to transfer angular momentum of the Earth to the Moon. This is one of several effects that may calendar changes unpredictably of the scale of a few thousand years of the dissipation by tides depends somewhat on sealevel and hence the presence of continental glaciers.

In general, the planets need to be represented as finite rotating bodies. Doing so, explained the precession of equinoxes with had been observed since antiquity and by the

Mayans. Beginning with data, the Earth's rotational axis processes like a slowly spinning top (Figure 8). Its effects can be viewed in two equivalent ways. The position of the pole of celestial sphere changes very slowly with time. In a few thousand years, Polaris will no longer be a good polestar. Equivalently, the position on the Zodiac of the Sun at equinox (when it is lined up with the Earth's geographic equator) changes slowly with time, giving rise to the modern name of precession of equinoxes. (See Primer on Seasons in Chapter 2.) The Sun at the spring equinox is now in Pisces. It was in Aries during antiquity. Ptolemy (ca. 85-165) attributed this to en masse movement of the stars in the sky.

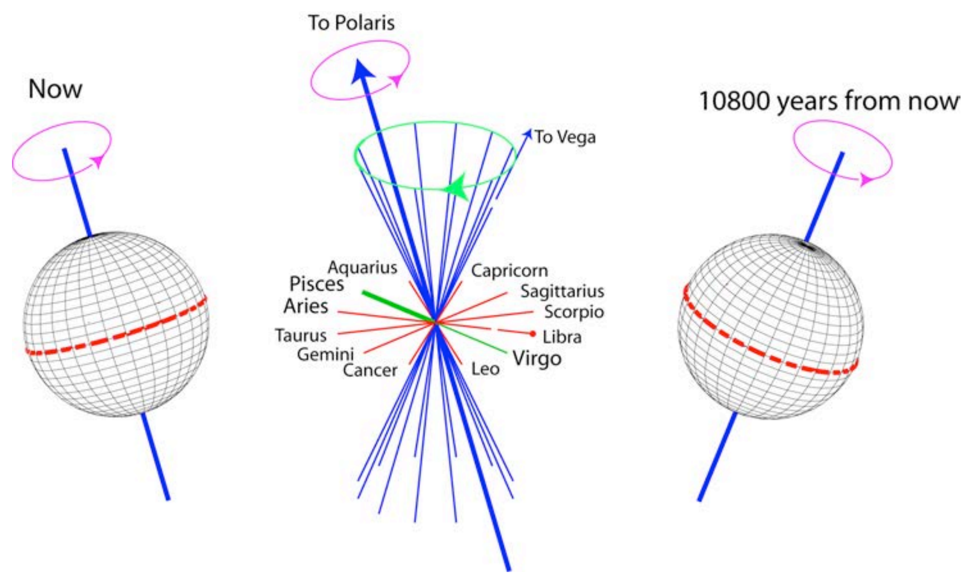


Figure 8: The geometry of the precession of equinoxes viewed from space. The position of the pole of the Earth's axis moves clockwise when view from the north. The Earth's orbit remains on the plane of the Ecliptic and the Sun continues to appear to pass from the Zodiac. The current northern spring equinox is in Pisces. In general, it is the direction on the plane of the Elliptic perpendicular to the axis. In 10,800 years the spring equinox will be in Libra and Vega will be the polestar. Precession does not reposition of the Earth's axis (north pole) relative to geographic places.

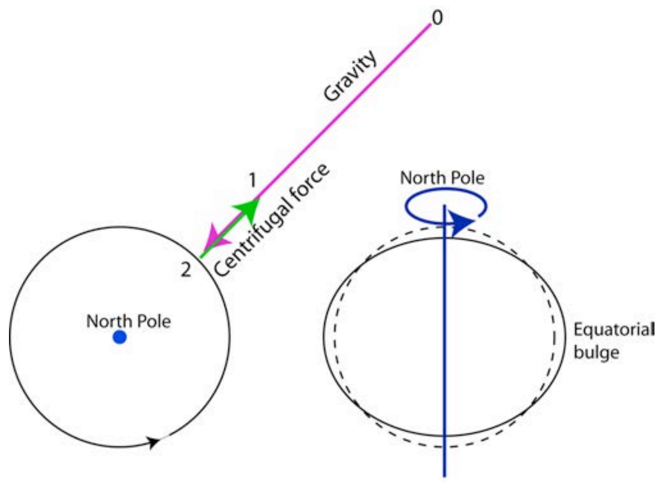


Figure 9: The equator viewed looking down from the North Pole (left). The counterclockwise motion of the Earth produces a centrifugal force on an object on its surface. The magnitude of this upward force (magnitude drawn from 2 to 1) is much smaller than the force of gravity (magnitude from 0 to 2). An observer on the surface senses the difference between these forces (magnitude from 2 to 1). The radius of the Earth is slightly larger at the equator than at the poles (right). Measurement of the effects in the late 1700s provided belated evidence that the Earth in fact rotates. The centrifugal force and the bulge are greatly exaggerated to make them visible.

Once Newton examined precession, he got a physical explanation in terms of the forces on a spinning object. The rotation of the Earth causes its radius to be larger at the equator than at the poles. Physically, objects on the surface of the Earth experience centrifugal force from the rotation. The effect is small enough that the qualms from antiquity that we would be thrown off a rotating Earth do not come into play. The force from rotation counteracts only about 1/288 of the Earth's gravitation at the equator. (Galileo approximately obtained this value for the Earth's surface.) In detail, the effect is more complicated because the bulge takes the Earth's surface away from the center of mass and because the bulge has a gravitational attraction. The acceleration from gravity is at the equator is about 0.5% less than that at the poles. This amount is easily resolved by instruments and satellite orbits but would not be evident in an athletic competition. For example, the distance of a golf drive would vary by a meter.

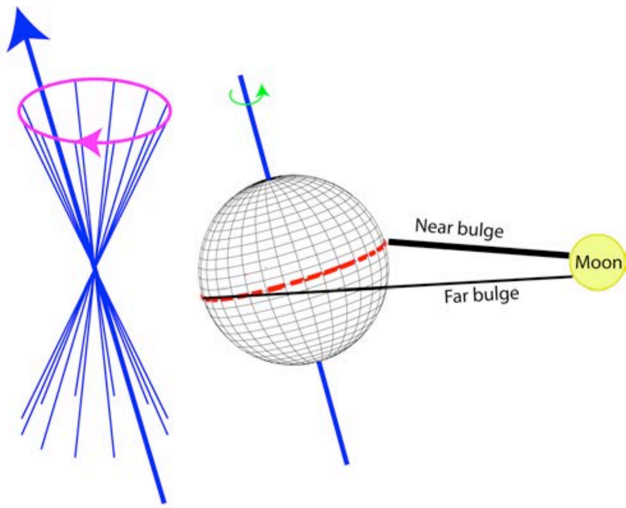


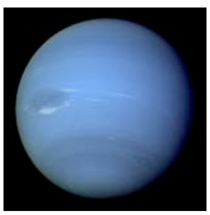
Figure 9: The rotational axis of the Earth precesses clockwise when view looking down on the North Pole (left). The Moon (or Sun) exerts more gravitational force on the near equatorial bulge than the far one. This acts with a sense to straighten the axis causing the axis to precess. Not to scale.

Precession results from the gravitational pull of the Sun and the Moon on the bulge. I draw only the Moon and the Earth. The Moon pulls harder on the near side of the bulge than the far side, which has the sense to lessen the obliquity of the Earth's axis. Precession viewed looking down on the North Pole is clockwise, opposite the spin. The celestial pole makes a clockwise track across the sky. The period is about 25,700 years. Note that gravity attempts to increase the tilt of a spinning top away from vertical, rather than decrease it. You can easily confirm that the precession of a top is in the same direction as its spin.

Further analysis of the gravitational forces in the solar system showed additional motions of the Earth. The obliquity of the Earth's axis with respect to its orbital plane (the ecliptic) changes with time. It is currently decreasing at a rate where the Arctic

Circle moves north by about a kilometer per century. The orientation of the Earth's orbit (the ecliptic) relative to the stars and the deviation of the shape of the orbit from a circle (ellipticity) also vary with time. This movement of the Earth's axis from precession is also irregular because both the Sun and the Moon exert significant gravitational pull. Astronomers call this effect nutation. This term distinguishes it from the medieval term for irregularities in precession, trepidation, which was the result of errors in the measurements.

These effects conclusively show that the Earth rotates on its axis. Astronomers have since Newton included these cumbersome processes in their predictions.



D3.3: Neptune NASA photo P-34611C by Voyager 2

Neptune, Mercury, and general relativity. Historically, there where two other serious problems with planetary predictions that led to major discoveries. Once the planet Uranus was discovered, astronomers just could not predict its orbit right. The answer turned out to be that the planet Neptune exists still farther away from the Sun. Two mathematical astronomers independently predicted its position using Newton's laws. They were right on. Neither could astronomers get Mercury's orbit right. The blinding search for other undiscovered planets near the Sun came up fruitless. This aided the discovery of the theory of general relativity.

General relativity (including Newtonian physics when relativistic effects are too small to worry about as with most daily situations on the Earth) is the basis of all mechanics. For astrobiology, this includes the behavior of stars (Chapter 6), climate and meteorology

(Chapter 7), the dynamics of the interior of planets (Chapter 8), and asteroid impacts (Chapter 11). The difference between Newtonian mechanics and general relativity is evident in careful tracking of space probes. Engineers use mechanics to control their velocities.

Bottom line: Please keep things simple!

The non-mathematically inclined reader will probably already be complaining about equations by this point. Still, Newton's laws are simple. There are only seven free parameters for each planet at the starting time of the calculation, 3 for its position, 3 for its velocity, and 1 for its mass. Once we have these for all the planets, we can compute the positions of the planets at any time in the future or even the past.

That is, we cannot only make testable predictions about the future, but we can examine data from antiquity. If one discovers another object, like Halley's comet, one can compute its orbit and show that it is the same comet that was seen many times in the past and predict its future arrivals. Astronomers have computed orbits with thousands of asteroids, comets, and spacecraft. They routinely model the orbits of multiple star systems and extrasolar planets.

We have seen a progression from epicycles to general relativity. This may not look like a simplification but it is. The number of rules and special restrictions decreased with each step and the number of possible predictions increased. One can still use epicycles to represent motions. (For the mathematical, epicycles are a form of infinite series.) Ptolemaic astronomy had special rules for each planet and made zero general predictions, like those that pointed to Neptune. From then on each step retained the previous step as a useful approximation. Kepler's laws are the case of Newton's laws where all the mass is in the Sun and the planets are small point masses. Newton's laws are the special case of general relatively where the velocities are low and the masses small. General relativity removed the restriction, known from the time of Newton, that Newton's laws do not apply to an accelerating coordinate system.

Conversely, modern scientists make simplifying assumptions when appraising hypotheses, like assuming that all the mass of the solar system is in the Sun, to obtain easily visualized results, like Kepler's laws. They then can see if the expected magnitude of an effect is big enough to investigate and whether the hypothesis grossly matches observations. They add detail if the hypothesis passes these tests.

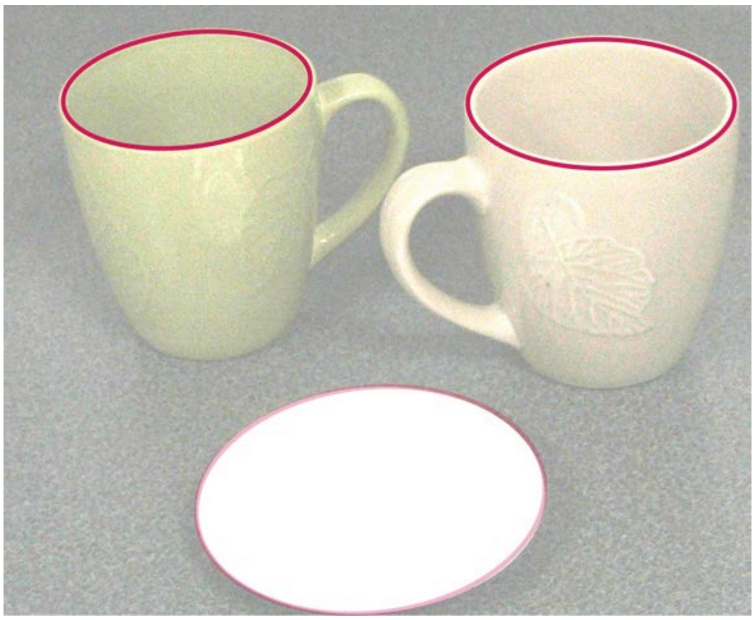
Good hypotheses sometimes make predictions that go against common sense. Scientists relish these situations as the hypothesis may fail and, if it passes, becomes much stronger. One famous prediction of general relativity is that light does not travel in a straight line, an example of the curvature of space around masses. This effect was confirmed by carefully observing the position of stars during an eclipse of the Sun. The calculations required considerable sophistication to get the actual very precise prediction, but the concept of curvature is simple enough to express in a single compact equation. In contrast, a vague pronouncement that something weird would happen during the eclipse is useless to science.

Overall, scientists crave simplicity. They call this philosophy Occam's razor, after William of Ockham (ca. 1284-ca. 1349), a Franciscan monk who died excommunicated. "The simpler explanation that represents the data should be preferred to the more complicated one." Modern scientists do not push this philosophy too far. It, like common sense, is a good guide but a poor master. For example, Kepler's ellipse is the simplest oval curve. Newton's exactly integer inverse square law is a lot simpler than say inverse 2.00001 power. Yet simplicity did not render these laws sacrosanct. Neither did their association with famous astronomers. Scientists relegated both laws to useful approximations when they proved wanting in the face of data. Physicists continue to test predictions of the theory of relativity. On the other hand, practicing scientists know from experience that a contrived and convoluted explanation like epicycles is likely to be wrong.

Primer on ellipses

Kepler selected the ellipse as his oval curve because of its mathematically simplicity. We have already seen in Figure 1 that an ellipse can be drawn with two tacks and a string. Greek mathematicians proved that cutting a cone or cylinder in two produces an ellipse. You can easily see this with cut carrots and sausages.

A circle viewed obliquely in also an ellipse. The reader who has some geometry can use this fact to show that (the inside of) the crescent on the moon is half an ellipse. This observation is the basis of the valid classical argument that the Moon is essentially spherical.



For those with some mathematics, an ellipse has a simple mathematical formula in 2 dimensions.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

where a and b are the lengths of the 2 axes. You can easily show that $a = b$ yields the formula for a circle.

Notes

The ancient concept of impetus arises from observations. Put simply, a projectile, like an arrow, acquires an impetus when it is launched. It continues in it original direction until its impetus becomes spent and then falls vertically to the ground. Force needs to be continually applied, like with an oxcart, to keep an object moving.

Newtonian physics recognizes air resistance as force that slows down projectiles, analogous to the friction of road and wheels that slows down oxcarts. Still the impetus description appears valid to the extent that you can causally observe when air resistance is high. Wad up and piece of paper and throw it. Baseballs, arrows, and golf balls encounter only moderate air resistance; they move with obviously arched trajectories, hence the term archer. Fielders learn by the time they are in Little League to judge the path of a fly ball. They are not always successful even at a professional level.

Impetus theory influenced educated medieval persons. Medieval paintings, for example, may show arrows falling vertically even though the uneducated archers and their targets on the battlefield knew that this did not happen.

Vestiges of this thinking persist in common speech, like with spent bullets. A quaint example applies to an animate object; a spent grasshopper is one that has fallen into a stream and has ceased its struggles. A spent grasshopper fly is an effective lure for large trout when cast along a grassy riverbank.

Uranus is visible to the naked eye by people with good eyesight on moonless nights. A systematic effort to chart dim stars would have led to its discovery in antiquity.

Neptune is visible in a small telescope. Galileo saw it but did not recognize it as a planet.

Exercises

Planetary dynamics and relativity are topics in mathematical physics. It takes years of training to do serious calculations. The first exercise lets the reader with some mathematics skills delve into the topic of planetary orbits. The second shows you that calculus surfaces everywhere but is typically not recognized as such. It will help you see that the intent of calculus is simple even though it requires training to apply it to most physical problems. The third is a practical application of the effect of the Earth's rotation on it surface gravity.

Planetary orbit calculation. The reader who is savvy with a scientific computer language like FORTRAN and elementary numerical analysis can get a quick idea of the difficulty of computing planetary orbits. Start with a tiny planet in a circular orbit about the Sun. Kepler's laws the give an "exact" solution to check your results. Work in two Cartesian dimensions.

What happens if you take large time steps?

Use polar coordinates. Does this help?

If you are familiar with more sophisticated methods of numerical analysis, you may try ways to make the Cartesian 2-D calculation work efficiently. Then you can move to 3 bodies in 3-D. This will usually generate complicated non-repeating orbits, especially if you make the masses similar.

Stealth calculus. Mathematics instruction that concentrates on the difficult aspects of evaluating derivatives and integrals typically begins in the 12^{th} grade or college. Only a minority actually receive instruction. This practice cloaks their common occurrence of derivatives and integrals in daily life and the simple intent of these concepts. People tend to avoid mathematical words, causing simple calculus to pass unrecognized. Here are some examples.

The numerical analysis method for determining planetary orbits is basically like the rules of a road rally or a treasure hunt. One is told to say go north 2 miles and then pick up another clue to go west 1 mile and so on. Anyone who can follow these rules can understand the intent of using calculus with numerical analysis to find the path of an object. Finding the area of a living room by dividing up into squares and counting them when buying a rug is an example of an integral to find area. Derivatives occur daily, for example, the slope of a road and the growth rate of a child. The monthly change in unemployment percentage and the like are regularly reported with other financial news. Such quantities are derivatives with respect to time. Economists discuss marginal quantities, like the increase in cost to a bakery when it makes 1 more donut. All marginal quantities are derivatives.

Find more examples of stealth derivatives, integrals, and numerical analysis that occur in daily life. Give the unit like centimeters (or inches) per year for the growth rate of a child. This will help you understand the intent and great utility of calculus.

Pendulum clock and gravity. The variation of gravity from equator to pole on the Earth is enough to affect the time on a pendulum clock. This precluded their use to determine absolute time during the age of exploration. Most pendulum clocks have a small adjustable mass on the pendulum to control its effective length and hence its period. If you have access to a pendulum clock, especially one with instructions, check out this mechanism. Do the instructions include explicit adjustments for latitude?